

Economic Models of Consumption

An Honors Thesis (Honors 499)

by

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Abstract:

This paper examines the evolution of consumption functions over time. It starts off with the Keynesian consumption function and then presents the problems and optimization methods that led to subsequent theories about consumption. Along the way the Lagrangian and Hamiltonian optimization methods are introduced, and the paper concludes with a discussion of the Ramsey-Cass-Koopmans model.

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Introduction: The Keynesian Consumption Function

Rulebooks can be found on a wide array of topics, but they are generally all read for the same reason. A good soccer player reads and learns exactly what they can and cannot do during a game in order to know the limits that they must follow, and perhaps to strategize as to how to get calls made against the opposing team. Similarly, a worker completing a project with various time constraints and a budget that must be followed would first aim to understand all of these constraints so that they could strategize as to what the best way to complete the project might be. Rulebooks, in most cases, are examined in order to figure out what limitations exist so that the best way to comply with them can be determined. Everyday activities are jam-packed with limitations, and one such activity concerns how much people buy, or their consumption. The constraints placed on consumption and how people deal with these limitations to get the most satisfaction possible is a topic widely investigated by economists.

Consumption constitutes the beginning and end of all economic activities. The end goal of consuming drives people to grow crops, work for companies, and create products so that these goods or services can be sold and exchanged for consumption. In the same way, all of these goods and services being produced are intended to eventually be consumed. Consumption affects the success and growth of any economy, so it can easily be seen why consumption is such an important topic. It is essential for economists to figure out which factors limit how much individuals consume, and how the consumption of individuals changes over time.

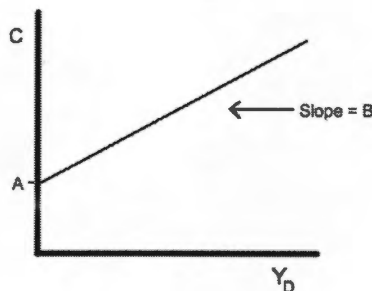
As we will see, there have been many different attempts at trying to figure out what causes people to buy things and in what amounts, but the first attempt that brought

attention to the issue of consumption was made by John Maynard Keynes in 1936 in his book *The General Theory of Employment, Interest and Money*. Keynes said that as people earn more income, their consumption rises. This suggests that consumption is made up of two parts; there is a base level amount that people will consume to buy necessities, like food, even if they have no income, and there is an amount that is proportional to a person's income so that if their income rises their consumption will rise as well. If this idea about consumption is written down in the form of an equation it will look like the following:

$$C = A + BY_D \quad (1.1)$$

In equation (1.1) the term A represents that base level of consumption (C) that will take place even when income is zero. B then represents what Keynes referred to as an individual's "marginal propensity to consume" which is how much a change in income causes consumption to change. For example, if income increased by \$1 and this caused consumption to increase by \$0.75 then the marginal propensity to consume would be 0.75. Finally, the Y_D term in the equation is disposable income, which is income after paying taxes because we are interested in how much income individuals can use for consumption, not pay taxes with. Equation (1.1) is a linear function, and if it were graphed it would look something like the following:

Figure 1.1



Keynes introduced this model of consumption purely based off of intuition and said that it could be deduced from “our knowledge of human nature and from the detailed facts of experience”(Keynes 96). As we will see, this method of determining a model for consumption differs tremendously from later methods where math is used to find the optimal level of consumption, and where data is often provided in support of the particular model.

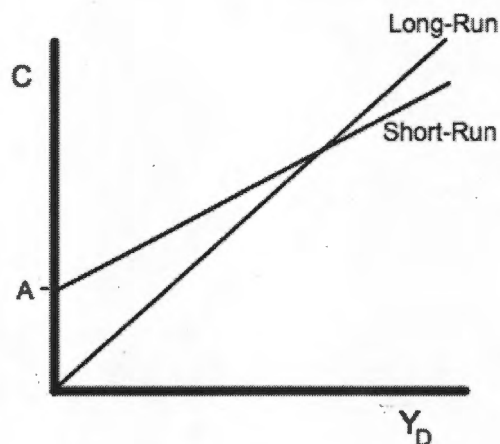
After Keynes proposed his model for consumption, people were soon analyzing the data to see if consumption during a certain time period really was closely related to disposable income in that same time period; to see if the data matched the model. While all of this data was being analyzed, someone named Simon Kuznets pointed out a contradiction to the former model in 1942 in his book *Uses of National Income in War and Peace*. Kuznets noticed that in the short run, the percentage of disposable income consumed decreases as income increases, due to the intercept term A being positive, which is consistent with the model proposed by Keynes. This idea is depicted in the chart below with fictional values chosen.

A	B	Y_D	C	Percentage of Y_D Consumed
15	0.65	30	34.5	115%
15	0.65	50	47.5	95%
15	0.65	70	60.5	86.4%

However, the data showed that the percentage of disposable income being consumed is very constant over the long run, implying that the value of A is very close to zero when considering long periods of time. This would suggest that a long-run

consumption model would look different than a short-run consumption model as depicted in *Figure 1.2*.

Figure 1.2



This finding was inconsistent with the assumptions made by Keynes, and has been called the Kuznets Paradox. After learning of the Kuznets Paradox, economists soon set out to try to explain it and came up with a number of interesting theories, but the most successful of these theories was the permanent-income hypothesis initially worked on by Brumberg and Modigliani and later expanded by Milton Friedman in 1957 (Parker 2-5).

The Permanent Income Hypothesis

Friedman said that income (Y) can be thought of as being made up of two separate components; there is permanent income (Y_P) and transitory income (Y_T) so that:

$$Y = Y_P + Y_T \quad (2.1)$$

Here permanent income is the average of what each consumer expects to earn for each remaining period of his or her life, in other words, it is the total amount that a consumer expects to earn during the rest of their life divided by the remaining number of time periods (such as years or fortnights) that they expect to live through. Transitory income is

then any positive or negative deviations from this level of permanent income. Transitory income results from unpredicted income from events such as an unexpected bonus, finding money on the sidewalk, or winning the lottery.

Consumption is made up of a permanent and a transitory part as well so that:

$$C = C_P + C_T \quad (2.2)$$

Where permanent consumption (C_P) is steady and planned based on permanent income and transitory consumption (C_T) is any positive or negative deviation from that steady level. About permanent consumption Friedman also said that it was proportional to permanent income so that:

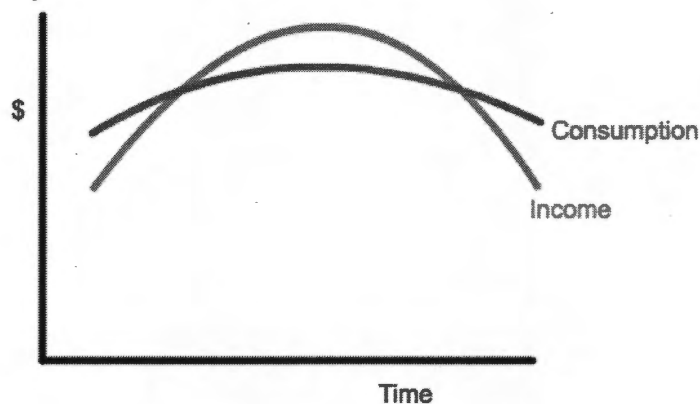
$$C_P = kY_P \quad (2.3)$$

Although temporary changes in consumption may be observed due to temporary changes in income, because permanent consumption is steady and based on permanent income, people generally will plan to even-out their consumption for each period of their life, and the value of k will be close to one (Friedman 26). We can see that with equation (2.3) the permanent income hypothesis answered the problem found in the Kuznets Paradox because it shows long-term permanent consumption increasing at the same percentage as permanent income as the two are said to be proportional.

Overall, without talking about any equations or whether something is permanent or transitory, the basis of the permanent income hypothesis is that people tend to smooth out their consumption over their life based on what they think their lifetime income will be. This is very different than the consumption function proposed by Keynes that only considered current disposable income. The permanent income hypothesis does not

necessarily predict that people will consume the exact same amount during each period of their life, but that generally they will smooth out their consumption based on what they expect to earn over their whole life rather than letting it vary drastically between time periods if income in those time periods varies. This is depicted in *Figure 2.1*.

Figure 2.1



By assuming that individuals smooth out their consumption completely over their lifetimes based on their permanent income, we can come up with a useful equation for what a person's consumption for each period would be. We said before that permanent income is the total amount that a consumer expects to earn during the rest of their life divided by the remaining number of time periods that they expect to live through, and if individuals are smoothing out their consumption completely, their consumption during any time period will equal their permanent income. If we let C_t and Y_t be an individual's consumption and income respectively during any time period t , suppose that they have a starting amount of wealth saved up called A_0 , and say that they will continue to live from $t = 1$ until $t = T$, then by this assumption their consumption is given by the following equation:

$$C_t = \frac{1}{T} (A_0 + \sum_{t=1}^T Y_t) \quad (2.4)$$

In this equation $(A_0 + \sum_{t=1}^T Y_t)$ is the total income that the consumer will get for the rest of their life because A_0 is the amount they start with and $\sum_{t=1}^T Y_t$ is their income from each remaining period all added together. Then because T is the total number of remaining time periods in the individual's life, this equation says that the amount they will consume in any given period is their total income for the rest of their life divided evenly among the remaining periods in their life.

By providing important observations on the topic of consumption, the permanent income hypothesis gives us some valuable insights into how the general public might act in response to government actions such as a tax break. Suppose that before any government action an individual earns \$20,000 in the first period of their life, \$60,000 in the second, and \$20,000 in the last; they start off with no initial wealth. According to equation (2.4) their consumption in any period should equal $\frac{1}{3}(\$0 + \$20,000 + \$60,000 + \$20,000)$, which equals \$33,333. This situation is depicted below, and it will be used as a starting point to compare how consumers might react to certain government actions.

Original Income and Consumption

	Period 1	Period 2	Period 3
Income	\$20,000	\$60,000	20,000
Consumption	\$33,333	\$33,333	\$33,333

Now suppose that the government implements some policy that raises income by \$10,000, and is known to be temporary so that it only affects income in the first period.

Then, according to equation (2.4), consumption in any period should equal

$\frac{1}{3}(\$0 + \$30,000 + \$60,000 + \$20,000)$, which is \$36,667.

Temporary Policy

	Period 1	Period 2	Period 3
Income	\$30,000	\$60,000	\$20,000
Consumption	\$36,667	\$36,667	\$36,667

Suppose that the government implements the same policy, only it is known to be permanent this time, so that income in every period is increased by \$10,000. Consumption in any period should then equal $\frac{1}{3}(\$0 + \$30,000 + \$70,000 + \$30,000)$, which is \$43,333.

Permanent Policy

	Period 1	Period 2	Period 3
Income	\$30,000	\$70,000	\$30,000
Consumption	\$43,333	\$43,333	\$43,333

From the above two tables showing the results of both a temporary and permanent version of a policy, it can be seen that both versions of the policy raise income in the first period by the same amount. However, the temporary version of the policy does not raise consumption in the first period as much as the permanent version does because the consumer spreads out the benefits from the temporary policy over their entire lifetime.

Consumers may not always know for sure whether a government policy will be permanent or temporary. Suppose the government implements a temporary policy that will raise income in the first period by \$10,000, but the consumer believes this policy to be permanent and thinks it will raise income in all periods. In the first period, they would determine how much to consume under the assumption that the policy will raise their

income in all periods, and so will consume $\frac{1}{3}(\$0 + \$30,000 + \$70,000 + 30,000)$, which is \$43,333. In consuming this much during the first period they are spending more than their income, and so will have to borrow \$13,333 that must be repaid later (we are ignoring interest rates). However, during the second period they will realize that the policy is only temporary, and will have to adjust their level of consumption. For the two remaining periods, consumption should equal $\frac{1}{2}(-\$13,333 + \$60,000 + \$20,000)$, which is approximately \$33,333. This pattern of income and consumption is shown in the following table.

Temporary Policy Believed to be Permanent

	Period 1	Period 2	Period 3
Income	\$30,000	\$60,000	\$20,000
Consumption	\$43,333	\$33,333	\$33,333

From this example of the temporary policy that is believed to be permanent, it can be seen that a lack of knowledge or misinformation about a situation can lead to an individual spreading out their consumption in a way that may not be in their best interest. The effect of this policy on consumption in the first period is exactly the same as the policy that is known to be permanent, even though it is temporary. This tells us that it is consumer's expectations about whether a policy is temporary or permanent that determines how they react, not whether the policy actually is permanent or temporary. This example shows that individuals' consumption habits can be different when they do not have all the information that they need.

Although the idea about consumption proposed in the permanent income hypothesis differs greatly from the consumption function proposed by Keynes, both of

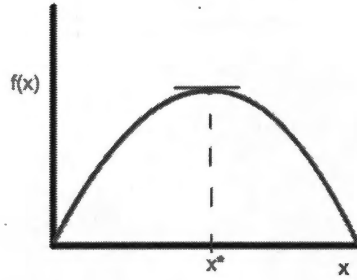
these ideas were developed and presented with relatively similar strategies. As was mentioned before, Keynes presented his consumption function simply by saying that it was based on common sense and knowledge of human nature. Friedman on the other hand talks a lot about how the permanent income hypothesis idea fits the data and how it can be tested, but the idea itself is presented with something more along the lines of an intuitive argument.

We will find that this strategy of making intuitive arguments and seeing if they fit the available data is not found in many subsequent models of consumption. Instead, many later models of consumption, although they are probably based on intuitive arguments, are presented with much more math, generally in the form of an optimization problem. These optimization problems use ideas from microeconomics, and in consumption models the goal of these problems is to maximize the utility, or satisfaction, that a consumer gets from consumption while dealing with their limited resources. We will now look closer at what these optimization problems entail, and how a relatively simple optimization problem can lead us to something called the neoclassical consumption model.

Introduction to Optimization and the Neoclassical Consumption Model

Suppose we are given a function $f(x)$ that looks something like that in *Figure 3.1*, and suppose that we wish to find the x -value where this function reaches a maximum:

Figure 3.1



From this graph it is easy to see that the function reaches its maximum where $x=x^*$. At this maximum point, $f(x)$ flattens and has a slope of zero. Thus, to find the value of x where $f(x)$ reaches its maximum, we must find the derivative of $f(x)$, set it equal to zero, and finally solve for x . This condition that the derivative must equal zero at the optimal point is only one of the conditions that must be fulfilled, but for now we will assume that these other conditions are accomplished. To be sure that this process is clear, an example of this type of problem is provided below.

Example:

Find the value of x where the function $f(x) = -(x - 2)^2 + 4$ reaches a maximum.

Step 1: Find the Derivative of the Equation.

Using the chain rule: $f'(x) = -2(x - 2)$

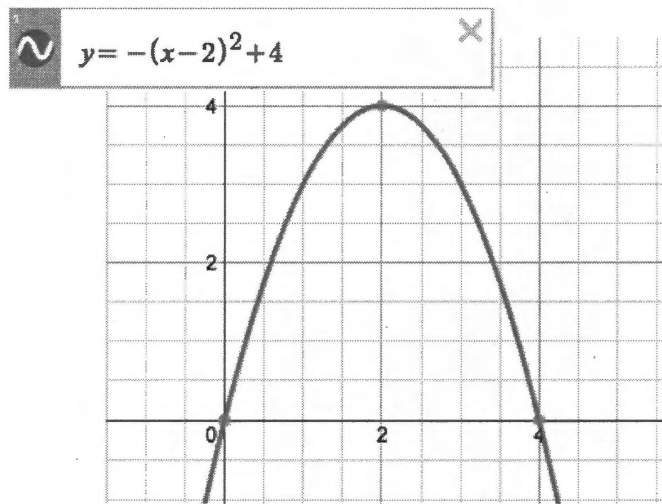
Step 2: Set the derivative equal to zero

Step 3: Solve for x .

$$0 = -2(x - 2)$$

$$x = 2$$

This tells us that $f(x)$ reaches a maximum when $x=2$. If we graph the function we see that this is indeed the case.



This example of finding the point where a function is maximized will be helpful to us when our goal is to choose consumption so that utility is maximized. For simple problems of this nature, the same method of taking a function's derivative and setting it equal to zero will be used. Before proceeding any further, it should be mentioned that this same method can be used to find a function's minimum, and if a function reaches a relative maximum at more than one point, it does not initially tell us which is the greater maximum point. Thus, problems could arise so that when we find where a function's derivative equals zero, our result is not the maximum point desired. We will not encounter problems of this nature here, and so for now we will assume that they do not exist.

Now that a basis has been provided for solving relatively simple optimization problems, we can use this knowledge to examine what is known as the neoclassical model of consumption. To do this we will first suppose that we are faced with a consumer that wants to spend everything they earn in two time periods with no prior savings in order to maximize their satisfaction. Yes, sadly this consumer knows they will die after two periods of time. They also know exactly how much they will earn, and what the interest rate will be.

When determining how much to consume over two periods, the thing that our consumer wants to maximize is the total utility that they get from all of their consumption in the two periods. This total utility will be all the utility from consumption in period one plus all of the utility from consumption in period two. Letting Z be total utility, U_1 be utility from consumption in period one, and U_2 be utility from consumption in period two, total utility is represented by the following:

$$Z = U_1 + U_2 \quad (3.1)$$

Because the consumer only will receive so much income to spend on consumption, they must face the following budget constraint:

$$C_2 = Y_2 + (1 + R)(Y_1 - C_1) \quad (3.2)$$

In this equation C_1 is consumption in period one, C_2 is consumption in period two, Y_1 and Y_2 equal the income for periods one and two, and R is the interest rate of borrowing or saving. Here $(1 + R)(Y_1 - C_1)$ is any money not spent from what was earned in period one plus interest earned on this amount. If more was spent in period one than was earned, the term $(Y_1 - C_1)$ becomes negative and is the size of the loan that must be taken out to allow the extra consumption above what was earned. In the case of a loan this constraint says that consumption in the second period must equal income in that period minus the loan from the first period and any interest that must be repaid. The budget constraint then simply says that in period two the consumer can spend only what they earned in period two plus or minus any savings or debt from period one.

In accordance with the problem we have seen previously, in order to find the point of optimal trade-off between consumption in periods one and two we begin by differentiating the function that consists of what we wish to maximize. This is equation

(3.1) because it is a function for total utility, and to differentiate it we will need to use the chain rule because the value of C_2 depends on C_1 . It can be seen through the budget constraint (3.2) that once C_1 is determined C_2 is also determined. Differentiating (3.1) using the chain rule leaves us with:

$$\frac{dZ}{dC_1} = \frac{dU_1}{dC_1} + \frac{dU_2}{dC_2} \left(\frac{dC_2}{dC_1} \right) \quad (3.3)$$

We then rewrite (3.3) with the knowledge that $\frac{dU}{dC}$ equals the marginal utility from consumption, which can be written as mu .

$$\frac{dZ}{dC_1} = mu_1 + mu_2 \left(\frac{dC_2}{dC_1} \right) \quad (3.4)$$

By looking at the consumer's budget constraint, it will be helpful to determine exactly what

$\frac{dC_2}{dC_1}$ is in order to simplify the above expression. The term $\frac{dC_2}{dC_1}$ is just another way of saying

"the change in consumption in period two divided the change in consumption in period

one". To see how C_2 changes as C_1 changes we will repeat equation (3.2): $C_2 = Y_2 + (1 +$

$R)(Y_1 - C_1)$. It will be helpful to us to first imagine a situation where $C_1 = Y_1$ so that

following the budget constraint, $C_2 = Y_2$. Then imagine that C_1 increases by one unit so that

now $Y_1 - C_1 = -1$ and $C_2 = Y_2 + (1 + R)(-1)$. By increasing C_1 by one unit, C_2 changed by

$(1 + R)(-1)$, and so $\frac{dC_2}{dC_1} = (1 + R)(-1)$. This tells us that the cost of consuming one more

unit in period one costs $(1 + R)$ units of lower consumption in period two. Plugging this

result into equation (3.4) yields:

$$\frac{dZ}{dC_1} = mu_1 + mu_2(1 + R)(-1)$$

We now set this derivative equal to zero and rearrange it for simplicity.

$$0 = mu_1 + mu_2(1 + R)(-1)$$

$$-mu_1 = mu_2(1 + R)(-1)$$

$$-\frac{mu_1}{mu_2} = (1 + R)(-1)$$

$$\frac{mu_1}{mu_2} = 1 + R \quad (3.5)$$

Thus, in order for the consumer to maximize their utility from consumption, they must follow the rule $\frac{mu_1}{mu_2} = 1 + R$ (Jones 6). This answer to the neoclassical consumption model provided by equation (3.5) makes sense because consuming where $\frac{mu_1}{mu_2} = 1 + R$ means consuming where the rate at which the consumer is willing to trade off consuming now for consuming later is equal to the rate at which they can trade current consumption for future consumption at the particular interest rate for saving or borrowing.

The neoclassical model of consumption can be very useful to us in a number of ways. Although it may seem to be limiting that this model considers only two time periods, in a more general sense the first period could be thought of as the present and the second period could be thought of as any time in the future. Furthermore, this model provides a result that is consistent with the permanent income hypothesis. If it is assumed that the interest rate R is zero and the consumer has no preference between consumption in period one or two so that a given level of consumption yields the same amount of utility in both periods, then the rule given by (3.5) now is $\frac{mu_1}{mu_2} = 1$. In order for this to hold true, mu_1 must equal mu_2 , and because each additional unit of consumption yields the same amount of additional utility regardless of the time period, then under these assumptions, C_1 must equal C_2 . This result shows that the consumer considered in the neoclassical model smoothes out their consumption to maximize utility, just as the permanent income hypothesis suggests.

Although the neoclassical model of consumption is very useful, we will soon need to examine models that involve a potentially infinite number of time periods. It is easy to see how a consumer's budget constraint and the function for total utility that they wish to maximize can get chaotic when a great number of time periods are considered. To deal with this problem, we must deviate from our simple method of taking derivatives and setting them equal to zero, and familiarize ourselves with the method of Lagrange multipliers.

Mathematical Intermission: Lagrange Multiplier Method

The Lagrange multiplier method allows us to minimize or maximize a function such as $f(x, y, z)$, which is subject to a certain constraint, such as $g(x, y, z) = C$. To do this, we form a new function:

$$L(x, y, z, \lambda) = f(x, y, z) + \lambda(g(x, y, z) - C) \quad (4.1)$$

Here λ is some constant called a "Lagrange multiplier"; notice that the term we multiply λ by is actually equal to zero (because $g(x, y, z) = C$). We then take partial derivatives of L with respect to x, y, z , and λ , then set those partial derivatives equal to zero. We solve for the values of x, y, z , and λ using the partial derivatives, the resulting x, y , and z values are the values that will minimize or maximize the function (Stewart 956-963).

To show that this method of using Lagrange multipliers gives accurate results, the problem of the neoclassical consumption model will be revisited. Our goal will again be to maximize total utility (Z) from consumption in two time periods, $Z = U_1 + U_2$. The constraint again is $C_2 = Y_2 + (1 + R)(Y_1 - C_1)$. The Lagrangian of this problem is:

$$L(C_1, C_2, \lambda) = (U_1 + U_2) - \lambda(Y_2 + (1 + R)(Y_1 - C_1) - C_2), \text{ or}$$

$$L(C_1, C_2, \lambda) = (U_1 + U_2) - \lambda(Y_2 + Y_1 - C_1 + RY_1 - RC_1 - C_2)$$

The partial derivatives with respect to the three variables are then taken and set equal to zero.

$$\frac{\partial L}{\partial C_1} = mu_1 + \lambda + \lambda R = 0$$

$$mu_1 = -\lambda(1 + R) \quad (4.2)$$

$$\frac{\partial L}{\partial \lambda} = -(Y_2 + Y_1 - C_1 + RY_1 - RC_1 - C_2) = 0 \quad (4.3)$$

$$\frac{\partial L}{\partial C_2} = mu_2 + \lambda = 0, \text{ or}$$

$$mu_2 = -\lambda \quad (4.4)$$

Substituting the value for $-\lambda$ found in (4.4) into (4.2) leaves us with:

$$mu_1 = mu_2(1 + R)$$

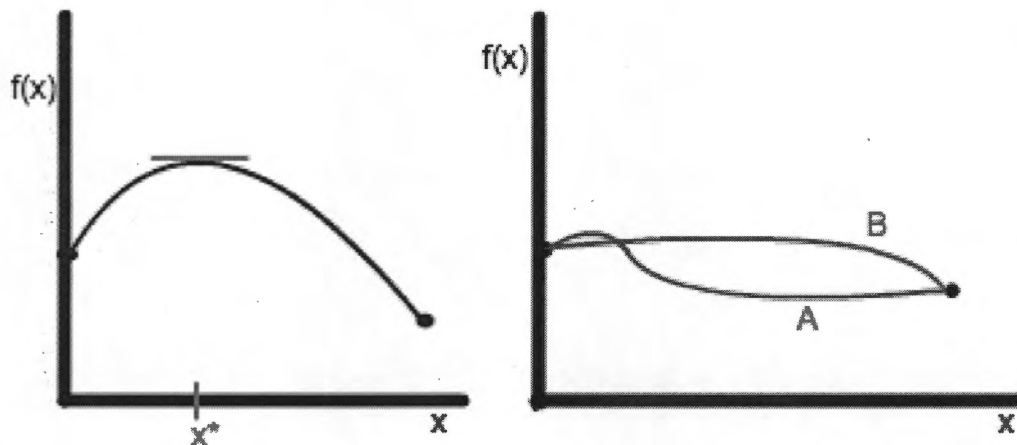
$$\frac{mu_1}{mu_2} = 1 + R \quad (4.5)$$

Notice that the result in (4.5) is exactly the same as what we found earlier in equation (3.5). One difference that may be noticed with this particular example is that while using the method with derivatives earlier we said that $\frac{\partial(U_1+U_2)}{\partial C_1}$ was $mu_1 + mu_2 \left(\frac{dC_2}{dC_1}\right)$ but in the example using the Lagrangian we said $\frac{\partial(U_1+U_2)}{\partial C_1}$ only equals mu_1 . This is because with the first method we only took the derivative with respect to C_1 , and C_2 was represented indirectly through the budget constraint. However, in the second method we choose to also take the partial derivative with respect to C_2 , so the chain rule is not necessary. Another way of seeing this is considering that we have the function $f(x) = 2x + (5x^3 + 1)^2$. We could take the derivative of this function only with respect to x using the chain rule to get $\frac{df}{dx} = 2 + 2(5x^3 + 1)(15x^2)$, or we could say that $(5x^3 + 1) = y$, and if it were convenient to us, we could differentiate the function with respect to x to get $\frac{\partial f}{\partial x} = 2$

and then also differentiate it with respect to y to get $\frac{\partial f}{\partial y} = 2y$. Thus, we see that the Lagrangian method can be relied upon to yield accurate results.

When first seeing the method of Lagrange multipliers, one might wonder how this method allows us to find a solution and what it is actually doing. Unlike the previous method, this method is used to maximize the value of an entire function for what can be called dynamic optimization problems. This idea of maximizing the value of a function can be shown in *Figure 4.1*.

Figure 4.1



The graph on the left side of the figure shows the previous idea of finding a point x^* where the function reaches a maximum. The graph on the right, however, shows that there could be multiple functions with the same starting and ending points, and one would want to make choices to maximize the total area under a function given some constraint. In *Figure 4.1* there are two options, option A and option B, and it can be seen that the area between the function and the x-axis is maximized when option B is chosen.

With this basic understanding of what we aim to do when solving dynamic optimization problems, how this Lagrangian method relates to another method called the

Hamiltonian and why that method works will be discussed later. For now the Lagrangian can be used to examine a number of problems that involve optimization over numerous time periods.

Revisiting the Permanent Income Hypothesis as an Optimization Problem

Earlier, we considered the permanent income hypothesis using what was essentially an intuitive argument. However, this hypothesis aims to explain a situation where consumers must maximize their utility from consumption over numerous periods while being constrained by their income. Now that we have an understanding of one such method that can be used to solve these types of problems, the permanent income hypothesis can be revisited in the form of an optimization problem.

First, suppose an individual knows that they will live for t separate periods of time (where t could be infinite) and they want to maximize their utility from consumption over their entire life. Of course the individual's consumption is constrained by their income, and they can only spend the wealth they start out with and any income that they earn in subsequent time periods. Although this problem of maximizing the individual's lifetime utility will be solved later using the method of Lagrange multipliers, we will first attempt to come to an intuitive explanation.

We will assume that the interest rate is zero and that there is no discount factor saying that individuals prefer consumption in earlier periods to consumption in later periods. This may seem like a rather large assumption, however, this is really no different than assuming that the interest rate and the rate at which future consumption is discounted are equal so that they cancel each other out; we will show why this is the case later. With this setup, one unit of consumption in any period yields the same utility as if

that consumption was saved until the next period. Also, we make the assumption that the utility function of each period shows diminishing marginal utility, so that each additional unit of consumption in any time period always yields less utility than the unit of consumption before it. Now assume that the consumer has kt units of total lifetime income, where k is any constant, which can be exchanged for kt units of consumption. Notice that kt can be evenly divided among the t time periods, so that k units of consumption take place in each period.

We claim that the consumer maximizes their utility by dividing their consumption evenly in such a way that each period contains k units of consumption. To show that this claim is true, assume that the consumer does not divide their consumption evenly among the time periods, so that one period contains $k + 1$ units while another contains $k - 1$. Because it was previously assumed that consumption in each period demonstrated diminishing marginal utility, the consumer must gain more utility from the k^{th} unit of consumption than from the $(k + 1)^{\text{th}}$ unit. Thus, the consumer can gain more utility by allocating a unit of consumption away from the period that contained $k + 1$ and toward the one only containing $k - 1$, so that each period contains exactly k units of consumption. This result provides the basis of the permanent income hypothesis where the individual maximizes their utility by considering their total lifetime income and spreading it out throughout their life. Now that we have looked at one simple way of viewing this problem, we can move on to considering it using the method of Lagrange multipliers.

By wishing to maximize the total utility from consumption throughout their life, our representative consumer would wish to maximize the function:

$$U = \sum_{t=1}^T U(C_t) \quad (5.1)$$

Here $U(C_t)$ stands for the utility from consumption in time period t , so the whole entire function is saying that we wish to maximize the total utility added up from each time period in the individual's life, from period 1 to the last time period T .

The individual whose utility we wish to maximize has a starting amount of wealth that will be called A_0 and they know they will earn income of Y_1, Y_2, \dots, Y_T for each respective time period of their life. If we say that they have to pay back any debt they have accumulated by the end of their life, their budget constraint becomes:

$$\sum_{t=1}^T C_t \leq A_0 + \sum_{t=1}^T Y_t \quad (5.2)$$

This constraint simply says that all of the individual's consumption added up over their whole lifetime cannot be greater than their initial wealth plus all of the income they earn during their whole lifetime. We know that the individual should seek to follow this budget constraint where the first part of the equation equals the second part because they will always gain utility by consuming more. Remember that our rule for the Lagrange multiplier method is to first set up the equation $L(x, y, z, \lambda) = f(x, y, z) + \lambda(g(x, y, z) - C)$ where $f(x, y, z)$ is the function that needs to be maximized and $g(x, y, z) = C$ is the constraint. So with the function with we wish to maximize (5.1) and budget constraint (5.2) the Lagrangian in this case is:

$$L = \sum_{t=1}^T U(C_t) + \lambda(A_0 + \sum_{t=1}^T Y_t - \sum_{t=1}^T C_t) \quad (5.3)$$

Taking the partial derivative of the Lagrangian with respect to C_t and setting it equal to zero results in:

$$\frac{\partial L}{\partial C_t} = U'(C_t) - \lambda = 0$$

$$U'(C_t) = \lambda \quad (5.4)$$

Without going any further with the other partial derivatives that are normally involved in this type of problem, we know that equation (5.4) must hold for utility to be maximized, and this equation is very insightful by itself. This equation says that the marginal utility gained from consumption in each time period must equal the same constant λ . In other words, the marginal utility from consumption in every time period must be the same. Because only the level of consumption can affect the marginal utility from consumption, this means that in order for the individual to maximize their lifetime utility, they must consume the same amount in each time period of their life so that $C_1=C_2.....=C_T$. If this is expressed using the budget constraint (5.2), consumption during any period must equal the following (Romer 366):

$$C_t = \frac{1}{T} (A_0 + \sum_{t=1}^T Y_t) \quad (5.5)$$

Notice that equation (5.5) is the exact same as equation (2.4), only (5.5) was reached from the result of an optimization problem. Equation (5.5) says that consumption in any period should be the total wealth the individual receives in their life divided equally among the number of periods in their life. Thus, we see that the results from this optimization problem are the same as the results reached from a more intuitive process that we saw earlier.

The Impact of Interest and Discount Rates

In the previous model we examined a case where a consumer wished to maximize their utility from consumption when their consumption was constrained by their income. We saw that utility was maximized when the consumer spread their consumption smoothly over each period of their life, and this would imply that borrowing or saving must take

place in periods of below average or above average income. However, in the previous model we chose to ignore the interest rate of borrowing or saving along with the discount rate. If a consumer is to spread their consumption out between each time period, there seems to be a high likelihood that they will need to borrow or save some of their income eventually, and because of this, it may be useful to include an interest rate of borrowing and saving in the model. The consumer also might value consumption in earlier periods more than consumption in later periods, perhaps because they may buy durable goods that they can use during subsequent periods, and so it would be appropriate to include a discount factor. The upcoming model will examine the same problem as the previous one, only it will include an interest rate and a discount rate.

As with the previous model, our goal is to maximize the utility gained from consumption, however, to be more precise, let's define the function $U(C_t)$ as:

$$U(C_t) = \frac{C_t^{1-G}}{1-G} \quad (6.1)$$

Here the value of G must be strictly between, not equal to, one and zero. This is a pretty interesting utility function where G can be called the "coefficient of relative risk aversion" (Romer 50). The smaller G is the more the consumer is willing to let their level of consumption vary over time. To see that this is the case consider two consumers each with utility function containing a different level of G . Say that for Consumer 1 the value of G is 0.3 and for Consumer 2 the value of G is 0.9. Both consumers plan on consuming 10 units in each of the next two periods of their life, for a total of 20 units per person in those two periods. Now suppose that both consumers are offered a deal by their local investor where they can give up 3 units of consumption in the first period (no borrowing allowed) and will

then receive 4 additional units of consumption in the second period. This situation, along with the utilities of each consumer, is depicted in *Figure 6.1*.

Figure 6.1

Situation When Choosing <u>Not</u> To Invest			
	<u>Period 1</u>	<u>Period 2</u>	<u>Total</u>
Consumption:	10	10	20
Utility (G=0.3):	7.160	7.160	14.320
Utility (G=0.9):	12.590	12.590	25.180
Situation When Choosing To Invest			
Consumption:	7	14	21
Utility (G=0.3):	5.578	9.061	14.639
Utility (G=0.9):	12.148	13.020	25.168

Here it is shown that Consumer 1, whose utility function has $G=0.3$, gains more utility by choosing to take the investor's deal and let their consumption vary from 7 units in the first period to 14 units in the second. However, Consumer 2, whose utility function has $G=0.9$, receives more utility when not taking the deal. Even though they would be consuming more over the two periods, Consumer 2 with the larger value of G is not as willing to let their level of consumption vary.

Given this utility function, we can include a discount factor in front of the utility function equal to $\frac{1}{(1+p)^t}$ here p is our discount factor; the greater p is, the more the consumer prefers consumption now to consumption later. The term $\frac{1}{(1+p)^t}$ then puts utility from consumption in terms of its present value, in other words, it discounts the value of consumption in the future into what it is worth at the present. Combining the utility function and discount factor, we wish to maximize the following:

$$U = \sum_{t=1}^T \frac{1}{(1+p)^t} \left(\frac{C_t^{1-G}}{1-G} \right) \quad (6.2)$$

The consumer's budget constraint will follow the same idea as before where the total they can spend cannot be greater than their initial wealth (A_0) plus the total income they receive throughout their life. However, because interest rates are now in the picture, this budget constraint must be rephrased. A certain amount of income that will be received t periods in the future can be put in terms of what its present value is by multiplying it by $\frac{1}{(1+R)^t}$ where R is the interest rate, so we will rephrase our budget constraint so that it says the present value of the consumer's total lifetime consumption cannot be greater than their initial wealth plus the present value of the income they will earn during the rest of their life. This revised constraint is written below (Romer 380):

$$\sum_{t=1}^T \frac{1}{(1+R)^t} C_t \leq A_0 + \sum_{t=1}^T \frac{1}{(1+R)^t} Y_t \quad (6.3)$$

Now that we have established the function where we wish to find the maximum (6.2) and a constraint (6.3) we can set up the Lagrangian as follows:

$$L = \sum_{t=1}^T \frac{1}{(1+p)^t} \left(\frac{C_t^{1-G}}{1-G} \right) + \lambda (A_0 + \sum_{t=1}^T \frac{1}{(1+R)^t} Y_t - \sum_{t=1}^T \frac{1}{(1+R)^t} C_t) \quad (6.4)$$

Taking the partial derivative of the Lagrangian with respect to C_t and setting it equal to zero results in:

$$\begin{aligned} \frac{\partial L}{\partial C_t} &= \frac{1}{(1+p)^t} C_t^{-G} - \lambda \frac{1}{(1+R)^t} = 0 \\ \frac{1}{(1+p)^t} C_t^{-G} &= \lambda \frac{1}{(1+R)^t} \end{aligned} \quad (6.5)$$

Without continuing with the other partial derivatives, the rule from equation (6.5) can be used, and we can define what λ must equal in this case so that (6.5) can be written in terms of variables found in (6.2) and (6.3). Rearranging (6.5) yields:

$$\lambda = \frac{(1+R)^t}{(1+p)^t} C_t^{-G}$$

Because λ is a constant, its value does not change no matter if $t = 1$ or $t = 72$. This means that the above equation can also be written as:

$$\lambda = \frac{(1+R)^{t+1}}{(1+p)^{t+1}} C_{t+1}^{-G} \quad (6.6)$$

Substituting this value of λ found in (6.6) into equation (6.5) and then doing some rearranging leaves us with the following (Romer 381):

$$\frac{1}{(1+p)^t} C_t^{-G} = \frac{(1+R)^{t+1}}{(1+p)^{t+1}} C_{t+1}^{-G} \frac{1}{(1+R)^t}$$

$$\frac{1}{(1+p)^t} C_t^{-G} = \frac{(1+R)^{t+1-t}}{(1+p)^{t+1}} C_{t+1}^{-G}$$

$$\frac{1}{(1+p)^t} C_t^{-G} = \frac{1+R}{(1+p)^{t+1}} C_{t+1}^{-G}$$

$$\frac{C_t^{-G}}{C_{t+1}^{-G}} = \frac{[(1+R)(1+p)^t]}{(1+p)^{t+1}}$$

$$\left(\frac{C_{t+1}}{C_t}\right)^G = \frac{1+R}{1+p}$$

$$\frac{C_{t+1}}{C_t} = \left(\frac{1+R}{1+p}\right)^{\frac{1}{G}}$$

$$C_{t+1} = C_t \left(\frac{1+R}{1+p}\right)^{\frac{1}{G}} \quad (6.7)$$

Equation (6.7) tells us how consumption in any period relates to consumption in the previous period depending on the discount and interest rate. If the interest rate (R) equals the discount rate (p), then the term $\frac{1+R}{1+p}$ equals one, making $\left(\frac{1+R}{1+p}\right)^{\frac{1}{G}}$ equal one so that $C_{t+1} = C_t$. This result is consistent with our previous model that did not include an interest or discount rate; when the effects of the interest and discount rate cancel each other out because the two are equal, consumption remains constant for each period. However, if the interest rate is greater than the discount rate, $\frac{1+R}{1+p}$ is greater than one, so that $\left(\frac{1+R}{1+p}\right)^{\frac{1}{G}}$ is

greater than one; this means that C_{t+1} must be greater than C_t and consumption is increasing over time. Similarly, if the interest rate is smaller than the discount rate, $\left(\frac{1+R}{1+p}\right)^{\frac{1}{g}}$ is less than one; meaning that C_{t+1} must be less than C_t and consumption is decreasing over time. By including an interest and discount rate into the model it is shown that although consumers still smooth out consumption over their lives, the interest rate encourages them to save more in earlier periods while the discount rate encourages them to spend more in earlier periods.

Consumption Under Uncertainty

Previously in our discussion of the permanent income hypothesis, and then when interest and discount rates were included, we saw how a consumer might plan out their consumption over their entire lifetime in order to maximize their satisfaction. The only problem is people generally do not know exactly how many time periods they have left in their life, what their utility function looks like, and typically how much they will earn during the remainder of their life. Because uncertainty with regards to the factors that affect consumption, such as income, is a common ailment, the way that consumption is predicted to behave under this uncertainty should be discussed.

In 1978 Robert Hall wrote an article titled "Stochastic Implications of the Life Cycle-Permanent Income Hypothesis: Theory and Evidence" arguing that because income cannot be predicted with certainty, the permanent income hypothesis implies that consumption follows a "random-walk". This idea that consumption cannot be predicted has been called the random-walk hypothesis.

The random-walk hypothesis is a response to the permanent income hypothesis and builds off of the result we found in equations (2.4) and (5.5), that $C_t = \frac{1}{T} (A_0 + \sum_{t=1}^T Y_t)$.

However, it is assumed that instead of knowing the total value of their remaining lifetime income besides their initial wealth, consumers can only have an expected value of this income, which is written as $\sum_{t=1}^T E[Y_t]$. When a consumer chooses how much to consume in any period, they base their decision off of what they expect their total future income to be as of that period. Under this assumption, consumption in any period can be written as:

$$C_t = \frac{1}{T} (A_0 + E_t[\sum_{t=1}^T Y_t]) \quad (7.1)$$

In (7.1) the subscript t after the E indicates that it is the expectation as of period t , and this expectation may change as time progresses. Equation (7.1) tells us basically the same thing as (2.4) and (5.5), that consumers try to spread out their consumption evenly among the T periods to maximize utility, only (7.1) also shows that consumption is subject to changing expectations.

The implication of (7.1) is that in any period the consumer adjusts their consumption so that the amount consumed in the following periods equals consumption in the current period. They do this based on their expectations in the current period in order to smooth out their consumption, which implies that changes in consumption are unpredictable because they are due to unpredictably changing expectations. The idea that consumption in any period equals the expectation as of the previous period of that consumption plus some error, denoted e_t , can be written as:

$$C_t = E_{t-1}[C_t] + e_t \quad (7.2)$$

In (7.2) e_t is any positive or negative difference between what C_t was expected to be in the previous period and what it actually turned out to be. Because we said earlier that the

expected value of consumption in the next period equals consumption in the current period, $E_{t-1}[C_t] = C_{t-1}$. This makes equation (7.2) equivalent to:

$$C_t = C_{t-1} + e_t \quad (7.3)$$

Equation (7.3) gives us the result of the random-walk hypothesis. It implies that the best estimate of future consumption is simply current consumption, and that deviations from the level of current consumption cannot be determined because the decision about current consumption is already made with all the relevant information in mind, thus deviations from the current level of consumption are random and cannot be predicted (Romer 373).

When Hall first published his idea of the random-walk hypothesis many economists thought it was a bunch of rubbish. Hall's idea differed greatly from the views held about consumption at the time, and many researchers set out to test its validity. Of the many tests conducted to either justify or disprove the random-walk hypothesis a good number came up with results contradicting the hypothesis, although it should be mentioned that there were many differences in opinion on what method to use and no test can claim to be perfect. Although many results have ran contrary to the random-walk hypothesis, the hypothesis does highlight the importance of realizing that future income cannot be perfectly certain, and therefore our best estimate of future consumption is only the level of current consumption.

One detail regarding the random-walk hypothesis that will lead into other considerations is that this hypothesis assumes that individuals act as if their expectations about future income are certain. In other words, they believe that their expectations as of period one are certain to be true, and then even if they turn out to be completely wrong, they believe that their revised expectations as of period two are now certain to be true.

Because it is assumed that individuals do not see a risk of their circumstances changing in the future, they do not engage in precautionary saving for unpredictable events. When individuals understand that there is a risk of their circumstances changing, they will likely participate in precautionary saving, and may look to acquire assets that reduce their risk. When consumers look for assets that will reduce their "risk" they prefer a specific characteristic in these assets, and to see what this characteristic is we must further look at this "risk" they are trying to minimize.

As consumers go about their lives they may be surprised with periods of unexpectedly low income, and for some reason or another may not be able to borrow enough to smooth out their consumption like they would prefer to. We will call these situations risky to consumers because in these circumstances they know that they are not getting the most utility possible and that they would be better off consuming more in the present. Thus a risky asset to consumers is one that has a high payoff when marginal utility of consumption is low (when consumption is already high), and has a low payoff when marginal utility is high (when consumption is low). Consumers would prefer an asset that has a high potential payoff when their consumption is low, and a low potential payoff when their consumption is already high so that they can use what they earn from the asset to smooth out their consumption. In other words, consumers view an asset as risky if it is unlikely to pay off when they want it to.

This idea that consumers value assets whose returns move inversely with their income over those that do not is the main idea behind what is called the consumption capital asset pricing model, or consumption CAPM. This model predicts that because consumers value those less risky assets more, they will be willing to pay more for these

assets and will require a lower rate of return from them. Interestingly, this argument suggests that individuals would not want to invest in the industry that they work in or in domestic companies because the payoff of these assets would likely be in some way positively correlated with their income (Romer 385).

Mathematical Intermission: The Basics of Optimal Control Theory

Until now we have examined a number of models that demonstrate how consumers go about making their consumption decisions. Now we will delve into more complex problems that consider more closely how the choices made about some variables influence the value of other variables that cannot be chosen directly. These problems that will be discussed consider continuous rather than discrete time, and although the Lagrangian method that was used in earlier problems will yield the same results, another method using what is called the Hamiltonian is typically used for problems of this nature that consider continuous time.

In types of problems where the Hamiltonian method is used, there are control variables and then there are state variables. If a variable can be chosen directly at any point in time it is a control variable, and if a variable is determined by past decisions about control variables, it is a state variable. To make these two types of variables clear, think about someone driving a car; at any point in time the driver can make decisions about their acceleration and direction, and so acceleration and direction can be thought of as control variables. The driver cannot, however, choose directly the distance they have traveled from their starting point. This value is determined by all the past decisions the driver has made about acceleration and direction, and so it is a state variable. Of course there can be other

variables affecting the movement of the car, such as weather and road conditions, that are beyond the driver's control and also must be taken into consideration.

Once the state and control variables have been identified, any constraints on these variables should be recognized. Suppose we have a control variable $c(t)$ and a state variable $s(t)$, because the value of $s(t)$ is determined by past decisions about $c(t)$, we should be able to find a constraint describing how the value of the state variable changes based on the value of the control variable. In this case our constraint might be $\Delta s(t) = g[c(t), s(t)]$. Along with any constraints, an objective function must be identified as well. An objective function is the function that we wish to maximize the value of; suppose our objective function is $f[c(t), s(t), t]$.

After identifying the state and control variables, and using this knowledge to recognize an objective function and any constraints, we can set up what is called the Hamiltonian. With an objective function $f[c(t), s(t), t]$ and constraint $g[c(t), s(t)]$ the Hamiltonian is defined as:

$$H = f[c(t), s(t), t] + \lambda(t)g[c(t), s(t)] \quad (8.1)$$

Here $\lambda(t)$ is sometimes called the co-state variable, and is defined as the marginal value, of the state variable $s(t)$, we will discuss why this is the case later.

The next step is to differentiate the Hamiltonian with respect to each control variable, in this case this is only $c(t)$, and set this derivative equal to zero. Also differentiate the Hamiltonian with respect to the state variable $s(t)$, and set it equal to $-\Delta\lambda(t)$ (Dorfman 822). This last step can be written as:

$$\frac{\partial H}{\partial c(t)} = 0 \quad (8.2)$$

$$\frac{\partial H}{\partial s(t)} = -\Delta\lambda(t) \quad (8.3)$$

These two equations (8.2) and (8.3) are the necessary conditions to maximize the objective function. If a maximum solution exists, then the rules found in these two equations must be followed.

We now know the steps that must be followed in order to solve an optimal control problem, but it has not been explained why this method of using the Hamiltonian works. The rest of this section discusses how the Lagrangian and Hamiltonian methods are related, and provides an example of an optimal control problem along with an explanation that was offered by Robert Dorfman as to why the Hamiltonian method works.

First we will consider how the Lagrangian can give us the same results as the Hamiltonian method. Consider the integral of our objective function, $\int f[c(t), s(t), t] dt$ as being made up of the sum of the values of the objective function over tiny time intervals of length dt . Then our objective function may also be written as $\sum f(c_t, s_t, t) dt$, and the constraint that was $\frac{ds}{dt} = g[c(t), s(t)]$ can now be written as $g(c_t, s_t) = (s_{t+dt} - s_t)/dt$, because $ds = (s_{t+dt} - s_t)$, so that $g(c_t, s_t)dt = (s_{t+dt} - s_t)$. If we write down the Lagrangian with this objective function and constraint we will get:

$$L = \sum f(c_t, s_t, t)dt + \sum \lambda_t(g(c_t, s_t)dt - (s_{t+dt} - s_t))$$

Then, as is usual with the Lagrangian method, we start taking partial derivatives and setting them equal to zero. Here we will just take those derivatives with respect to c_t and s_t ;

$$\frac{\partial L}{\partial c_t} = \frac{\partial f}{\partial c_t} dt + \frac{\lambda_t \partial g}{\partial c_t} dt = 0$$

$$\frac{\partial L}{\partial s_t} = \frac{\partial f}{\partial s_t} dt + \frac{\lambda_t \partial g}{\partial s_t} dt + \lambda_t - \lambda_{t-dt} = 0$$

It may not be easy to see where the last two terms $\lambda_t - \lambda_{t-dt}$ on the derivative with respect to s_t come from, so to see this more easily, remember that $\sum \lambda_t(g(c_t, s_t)dt - (s_{t+dt} - s_t))$ is a sum equaling $\lambda_0(g(c_0, s_0)dt - (s_{dt} - s_0)) \dots + \lambda_{t-dt}(g(c_{t-dt}, s_{t-dt})dt - (s_t - s_{t-dt})) + \lambda_t(g(c_t, s_t)dt - (s_{t+dt} - s_t))$. It should be noticed that s_t shows up both in above constraint for when time= $t - dt$ and when time= t .

Now if the above derivative taken with respect to c_t is divided by dt we are left with:

$$\frac{\partial f}{\partial c_t} + \frac{\lambda_t \partial g}{\partial c_t} = 0$$

This equation is the same as $\frac{\partial H}{\partial c_t}$, and the above condition is equivalent to equation (8.2). If the lambdas (λ) are moved to the left side of the derivative taken with respect to s_t , and then it is divided by dt we are left with:

$$\begin{aligned} \frac{\partial f}{\partial s_t} dt + \frac{\lambda_t \partial g}{\partial s_t} dt &= -\lambda_t + \lambda_{t-dt} \\ \frac{\partial f}{\partial s_t} + \frac{\lambda_t \partial g}{\partial s_t} &= -\frac{\lambda_t - \lambda_{t-dt}}{dt} \end{aligned}$$

The term on the right hand side of this equation is equivalent to $-\frac{d\lambda_t}{dt}$ because the numerator is simply the change in λ over the tiny interval of time dt . The term on the left is equivalent to $\frac{\partial H}{\partial s_t}$, and so from this equation we get the same result as in equation (8.3) (Lich-Tyler 58).

The Hamiltonian and Lagrangian look very similar when they are written in a general form. Remember that given an objective function $f(x, y, z)$ and constraint $g(x, y, z) = C$, the Lagrangian is written as: $L = f(x, y, z) + \lambda(g(x, y, z) - C)$. Given the same objective function and a constraint $j(x, y, z)$ the Hamiltonian can be written as $H = f(x, y, z) + \lambda(t)(j(x, y, z))$. Although these two functions are very similar, when they are in

general form like this two differences can be detected. First, in the Lagrangian λ is a constant, while in the Hamiltonian it is a function of time written as $\lambda(t)$. Second, in the Lagrangian we subtract the budget constraint from itself so that the term λ is multiplied by is equal to zero, while in the Hamiltonian this is not done. These two differences are easily spotted when the two functions are written generally, however, there is one major difference between the two functions that cannot be directly spotted when the two functions are written in this general way. It is important to understand that the constraint used in the Lagrangian $g(x, y, z) = C$ is not necessarily the same as the constraint $j(x, y, z)$ used in the Hamiltonian, and this is obviously hard to see when the two functions are written so generally. When determining what the constraint is for the Lagrangian one must think of what the limits to one's behavior are, such as how much can be spent over a lifetime with regards to lifetime income; this constraint is like drawing a line that cannot be crossed. However, the "constraint" for the Hamiltonian does not exactly define a boundary; it instead tells us how the state variable grows.

Now that we know how the Hamiltonian can be derived from the Lagrangian method and the difference in the constraint used for each, we will look at an example of an optimal control problem and provide an explanation of an account given by Dorfman as to why this method works. Consider a firm whose goal is to maximize their profits, and that makes decisions, $d(t)$, at every point in time that affect profits. These decisions also affect the stock of capital, $k(t)$, available to them so that $\Delta k(t) = f(k(t), d(t), t)$, and the stock of capital is a factor that determines profits as well. In this situation, $d(t)$ is the control variable because it is chosen directly, and $k(t)$ is the state variable because it is determined indirectly by past decisions regarding $d(t)$. Profits earned at each point in time are a

function of $d(t)$ and $k(t)$, and so profits per unit of time can be written as $u(k(t), d(t), t)$. If we consider profits from when $t = 0$ until $t = T$, then the value that we wish to maximize is $\int_0^T u(k(t), d(t), t)dt$, which is total profits during that time, and so this is the objective function.

With the goal of maximizing the entire function of profits per unit of time, the best possible time path of $d(t)$ would need to be determined. Rather than trying to tackle this problem at face value, a better strategy is to try to simplify it so that it resembles the problem of trying to find a single maximum point. To do this, we can first break up the function we wish to maximize, which is $\int_0^T u(k(t), d(t), t)dt$, into two parts; one part consists of an extremely short time interval, Δ , that is so short that decisions made, $d(t)$, cannot be altered. By doing this, $\int_0^T u(k(t), d(t), t)dt = u(k(t), d(t), t)\Delta + \int_{\Delta}^T u(k(t), d(t), t)dt$. It is important to realize that although $d(t)$ remains constant during the time interval Δ , $k(t)$ continues to grow at a rate dependent on that value of $d(t)$. Again using the example of someone driving a car, in a period of time so short that the driver cannot change direction, the distance the car travels can still increase based on that direction.

Suppose some policy for determining the value of $d(t)$ is followed during the time interval Δ , and afterward the best possible policy is determined until $t = T$. Under this assumption, we only have to choose the best policy during Δ , and because $d(t)$ cannot change during the extremely short time-span of Δ , we just need to find the one best value for $d(t)$. During the time interval Δ , $d(t)$ must serve the purpose of increasing profits and growing the accumulated amount of capital. We then wish to maximize the contribution of

$d(t)$ to profits during this period, $u(k(t), d(t), t)$, and the value of capital stock. If we allow $\lambda(t)$ to equal the marginal value of capital at time t , then the value of capital stock accumulated during Δ is the capital stock accumulated, $f(k(t), d(t), t)$ multiplied by its value $\lambda(t)$, and this is $\lambda(t)f(k(t), d(t), t)$. So, by wishing to maximize the profits earned and the value of capital accumulated during the interval Δ , we wish to maximize $u(k(t), d(t), t) + \lambda(t)f(k(t), d(t), t)$, which just so happens to be the definition of the Hamiltonian in this case. $H = u(k(t), d(t), t) + \lambda(t)f(k(t), d(t), t)$.

Now that the Hamiltonian, which describes more precisely what we wish to maximize, has been established, we must go about actually maximizing it. Because none of the control variables can change during the short time interval Δ , we want to find the single value of $d(t)$ where H reaches a maximum. This problem has been transformed into one of the much simpler problems we have seen before of finding a single maximizing value, and so it can be solved in the same manner as these problems. We take the derivative of H with respect to our control variable $d(t)$ and set the derivative equal to zero, resulting in $\frac{\partial H}{\partial d(t)} = 0$, which is equivalent to equation (8.2).

Initially, it may seem as if we should take the derivative of the Hamiltonian with respect to $k(t)$, and set that equal to zero as well, but we must remember that $k(t)$ can change during the interval Δ , and so it is defined differently. To see clearly what needs to be done with respect to $k(t)$, rewrite the Hamiltonian remembering that $\Delta k(t) = f(k(t), d(t), t) = \frac{d}{dt} k$, so that $H = u(k(t), d(t), t) + \frac{d}{dt} \lambda(t)k(t)$. Using the product rule for taking derivatives, this can also be written as: $H = u(k(t), d(t), t) + \lambda(t)\Delta k(t) +$

$\Delta\lambda(t)k(t) = u(k(t), d(t), t) + \lambda(t)f(k(t), d(t), t) + \Delta\lambda(t)k(t)$. We now can take the derivative of this newly defined Hamiltonian with respect to $k(t)$, set it equal to zero, and get: $\frac{\partial H}{\partial k} = \frac{\partial u}{\partial k} + \lambda \frac{\partial f}{\partial k} + \Delta\lambda = 0$ (Dorfman 822). This gives us the result that $\frac{\partial H}{\partial k(t)} = -\Delta\lambda(t)$, which is equivalent to equation (8.3). Thus, by breaking our objective function into two parts, one part consisting of an extremely small time-interval Δ , and considering the impact of our control and state variable during the interval Δ , it can be seen why the Hamiltonian and resulting maximum conditions found in equations (8.2) and (8.3) can be used to solve optimal control problems.

The Ramsey-Cass-Koopmans Model

A popular model where the Hamiltonian is often used to arrive at a conclusion is the Ramsey-Cass-Koopmans model. This model considers how an entire group of people can make optimal consumption choices when these choices influence other factors of the economy, such as the amount of capital available. It is an economic growth model that uses optimal control theory to arrive at a conclusion as to how consumption should take place to maximize the total utility of everyone, and it is based on the work done by Frank Ramsey in 1928 and was built upon separately by Cass and Koopmans in 1965. One important detail about this model is that by determining the amount of consumption that takes place, it also implicitly determines the amount of saving that takes place as well because we say that savings is income minus consumption. This detail makes this model different than many other economic growth models, however, unlike some growth models that define a term they call "technological change" in terms of variables in the model, technological change (A) is exogenously determined in the Ramsey-Cass-Koopmans model.

The model considers the economy as being made up of identical households, so that one household can be representative of all of them. The goal is then to maximize the household's total utility from consumption. The representative household's utility function is:

$$U = \int_0^{\infty} e^{-pt} U(C(t)) \left(\frac{L(t)}{H} \right) dt \quad (9.1)$$

Here e^{-pt} is the discount factor where the bigger p gets the less the household values future consumption relative to current consumption. $U(C(t))$ is the utility from consumption where $C(t)$ is consumption per laborer at time t . This utility as a function of consumption will be defined as $U(C(t)) = \frac{C(t)^{1-G}}{1-G}$; this is the same utility function that we saw for consumption in the model examining the impact of the interest and discount rates, and again G shows how the household is willing to let its level of consumption vary over time. Finally, $L(t)$ is the number of total laborers in the economy, and H is the number of households, making $\frac{L(t)}{H}$ the number of laborers per household. Thus, the model wishes to maximize the total utility that each household gets from consumption from now until forever.

This model says that total income (Y) is a function of capital (K) and each effective worker. The amount of effective labor is written as AL where L stands for labor and A can stand for technological change but is anything that makes labor more productive. So, $Y = f(K, AL) = K^a(AL)^{1-a}$. The goal now is to write the function we wish to maximize, (9.1), so that consumption and labor are written in units per effective laborer; this will make the function easier to work with later. Let $c(t)$ be consumption per effective laborer. Because $C(t)$ is consumption per laborer, $c(t) = \frac{C(t)}{A(t)}$ and $C(t) = A(t)c(t)$. This makes the utility

function $U(C(t)) = \frac{C(t)^{1-G}}{1-G}$ equal to $\frac{(A(t)c(t))^{1-G}}{1-G}$. It is assumed that $A(t)$ grows at a constant

rate g , so that $A(t) = A(0)e^{gt}$ the function then becomes $\frac{(A(0)e^{gt})^{1-G} c(t)^{1-G}}{1-G} =$

$\frac{A(0)^{1-G} e^{(1-G)gt} c(t)^{1-G}}{1-G}$. Substituting this back into the original function and assuming that

labor grows at a constant rate n so that $L(t) = L(0)e^{nt}$ yields:

$$U = \int_0^\infty e^{-pt} \left(\frac{A(0)^{1-G} e^{(1-G)gt} c(t)^{1-G}}{1-G} \right) \left(\frac{L(0)e^{nt}}{H} \right) dt$$

Now factoring out the constant terms (those that don't depend on t) gives us:

$$U = A(0)^{1-G} \left(\frac{L(0)}{H} \right) \int_0^\infty e^{-pt} e^{(1-G)gt} e^{nt} \left(\frac{c(t)^{1-G}}{1-G} \right) dt$$

$$U = A(0)^{1-G} \left(\frac{L(0)}{H} \right) \int_0^\infty e^{(-p+(1-G)g+n)t} \left(\frac{c(t)^{1-G}}{1-G} \right) dt$$

Let $A(0)^{1-G} \left(\frac{L(0)}{H} \right) = B$ and let $\beta = p - n - (1-G)g > 0$ so that $-\beta = -p + (1-G)g +$

n . Finally, the former equation is now in terms of units per effective laborer and can be rewritten as:

$$U = B \int_0^\infty e^{-\beta t} \left(\frac{c(t)^{1-G}}{1-G} \right) dt \quad (9.2)$$

The amount of consumption that takes place is something that can be chosen directly by the household at each point in time, making $c(t)$ the control variable in this model. However, the total amount of consumption is limited by income, and this income is limited in part by the amount of capital available. The total amount of capital accumulated depends on the investment of firms, which depends upon the savings that takes place, and it was described previously how this savings depends on consumption. Because the stock of capital depends upon previous choices about the control variable consumption, and its value affects current and future income, it is the state variable. If we consider capital in

terms of capital per effective worker (k), such that $k = \frac{K}{AL}$, the change in k is equal to actual investment minus the amount of investment required to maintain the same level of k , called the break-even level of investment. Because $k = \frac{K}{AL}$, and A and L are growing at rates g and n respectively while K depreciates at a rate of δ , the break-even level of investment is $(n + g + \delta)k$ so that K keeps up with the growing amounts of A and L and makes up for its own depreciation. We will say that actual investment equals savings which equals income minus consumption, or $f(k) - c$. This leaves us with:

$$\Delta k = f(k) - c - (n + g + \delta)k \quad (9.3)$$

We now wish to maximize (9.2) with the rule (9.3). The Hamiltonian for this problem is then:

$$H = B \int_0^\infty e^{-\beta t} \left(\frac{c(t)^{1-G}}{1-G} \right) + \lambda(t)(f(k) - c(t) - (n + g + \delta)k) \quad (9.4)$$

By differentiating the Hamiltonian with respect to our control variable, $c(t)$, and setting this equal to zero we get:

$$\begin{aligned} \frac{\partial H}{\partial c(t)} &= B e^{-\beta t} c(t)^{-G} - \lambda(t) = 0 \\ B e^{-\beta t} c(t)^{-G} &= \lambda(t) \end{aligned} \quad (9.5)$$

We now need to differentiate the Hamiltonian with respect to our state variable, $k(t)$, and setting this equal to $-\Delta\lambda(t)$ we get:

$$\begin{aligned} \frac{\partial H}{\partial k} &= \lambda(t)[f'(k) - (n + g + \delta)] = -\Delta\lambda(t) \\ f'(k) - (n + g + \delta) &= -\frac{\Delta\lambda(t)}{\lambda(t)} \end{aligned} \quad (9.6)$$

From this point we can use equation (9.5), solve for what $-\frac{\Delta\lambda(t)}{\lambda(t)}$ must equal, and plug this value into (9.6). It is important to remember that when one takes the natural log

(\ln) of some variable, and then takes its derivative with respect to time, the result is that variable's growth rate. This is demonstrated below as we first take the natural log of both sides of (9.5), and then differentiate it with respect to t .

$$\ln B - \beta t - G[\ln c(t)] = \ln \lambda(t)$$

$$-\beta - G\left(\frac{\Delta c(t)}{c(t)}\right) = \frac{\Delta \lambda(t)}{\lambda(t)}$$

This value found for $\frac{\Delta \lambda(t)}{\lambda(t)}$ can now be plugged into equation (9.6), and we then simplify and rearrange it to solve for what the growth rate of $c(t)$ must be.

$$f'(k) - (n + g + \delta) = \beta + G\left(\frac{\Delta c(t)}{c(t)}\right)$$

$$f'(k) = n + g + \delta + p - n - (1 - G)g + G\left(\frac{\Delta c(t)}{c(t)}\right) \quad (\text{By plugging in } \beta = p - n - (1 - G)g)$$

$$f'(k) = g + \delta + p - (g - Gg) + G\left(\frac{\Delta c(t)}{c(t)}\right)$$

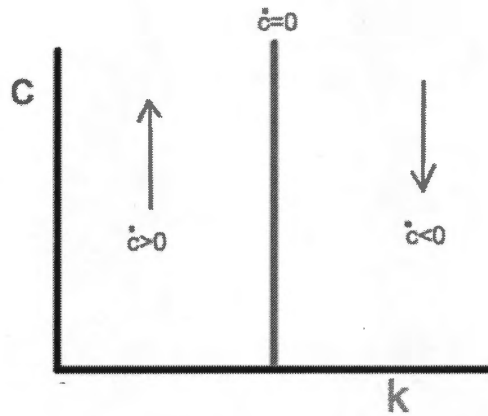
$$f'(k) = \delta + p + Gg + G\left(\frac{\Delta c(t)}{c(t)}\right)$$

$$f'(k) - \delta - p - Gg = G\left(\frac{\Delta c(t)}{c(t)}\right)$$

$$\frac{\Delta c(t)}{c(t)} = \frac{f'(k) - \delta - p - Gg}{G} \quad (9.7)$$

Equation (9.7) is our rule for how $c(t)$ must act over time to maximize total utility. To look at what this means for consumption and capital per effective laborer we can first see from this equation that if $f'(k) = \delta + p + Gg$, then the growth rate of $c(t)$ is zero. Call k^* the value of k that makes $f'(k) = \delta + p + Gg$, then before this value of k is reached, consumption per effective worker is increasing because $\frac{\Delta c(t)}{c(t)}$ is positive. After this value of k , consumption is decreasing because $\frac{\Delta c(t)}{c(t)}$ is negative, as depicted in *Figure 9.1*

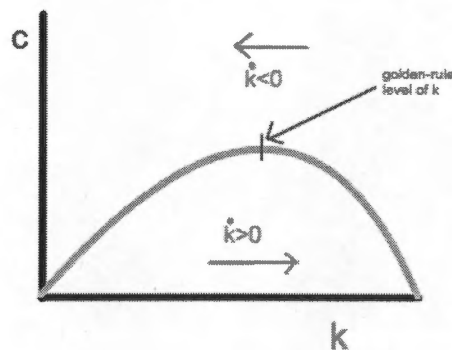
Figure 9.1



Here the dot above the c indicates "the change in $c(t)$ ".

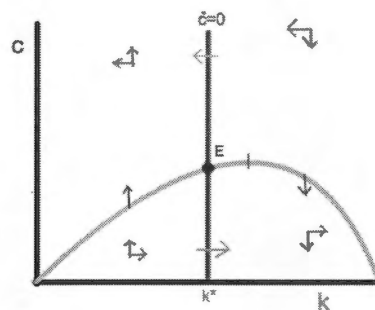
Similarly, we can find the value of $c(t)$ where the growth rate of k is zero. Because $\Delta k = f(k) - c - (n + g + \delta)k$ from (9.3), the change in k will equal zero if $c = f(k) - (n + g + \delta)k$. From the equation $c = f(k) - (n + g + \delta)k$, we can see that here $\frac{\Delta c}{\Delta k} = f'(k) - (n + g + \delta)$. This means that where k is not changing, c is increasing until it is maximized where $f'(k) = n + g + \delta$, and after that it is decreasing. It seems like a pretty beneficial thing to maximize consumption, and so $f'(k) = n + g + \delta$ is called the golden-rule level of k . This golden-rule point, along with the dynamics of k with respect to c , is depicted in Figure 9.2.

Figure 9.2



Here we see that above the parabola showing where k is not changing, k is decreasing, and below this parabola k is increasing. If we are at a point above the parabola, k is decreasing causing $f(k)$ to decrease, which should cause consumption to decrease, and so we move toward the parabola where k is constant. Similarly, if we are at a point below the parabola, k is increasing causing $f(k)$ to increase, which should cause consumption to increase, and so we move up toward the parabola where k is not changing. So it can be seen that the steady state here has to occur on the parabola showing where k is not changing. Now if we put the diagram showing the movement of c and the diagram showing the movement of k together we get what is shown in *Figure 9.3*.

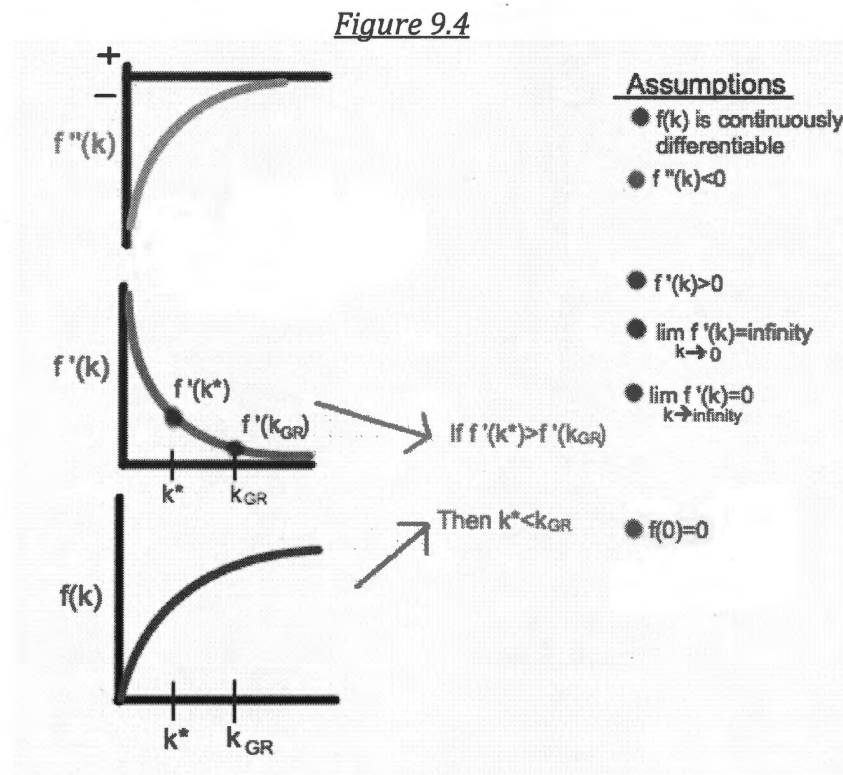
Figure 9.3



This diagram simply tells us what the other two told us where the arrows simultaneously show the movement of c and k at certain points. For example, in the upper left hand corner the set of arrows show us that at that point consumption per effective worker is increasing while capital per effective worker is decreasing. From this diagram we can see that the steady state, the point where neither c nor k is changing, is at the point labeled E .

One interesting thing about the location of the steady point E shown above is that it is to the left of the golden-rule level of capital, so it is not at the point where consumption per effective worker is maximized. We can show that E is to the left of this maximum point

if we can show that k^* is less than the golden-rule level of k , which we will call k_{GR} . For the model to show diminishing marginal productivity of k we have to assume that $f''(k)$ is negative. Assuming $f''(k)$ is negative means that $f'(k)$ has a negative slope, which means that $f(k)$ is increasing at a decreasing rate. So if $f'(k^*) > f'(k_{GR})$, then $k^* < k_{GR}$ and point E is near the left of the golden-rule point. This is depicted in *Figure 9.4*. The six assumptions that are made about $f(k)$ are known as the Inada conditions (Romer 12).

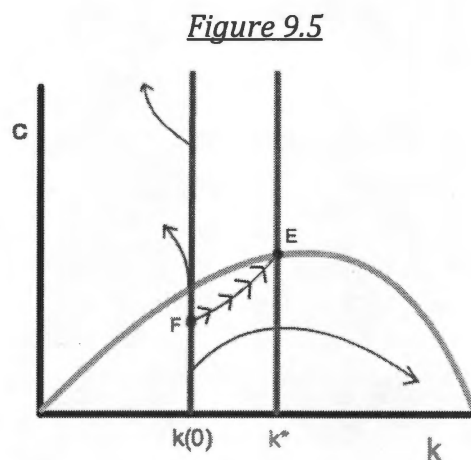


Now in order to show that E is to the left of the golden-rule point because $k^* < k_{GR}$, we need to show that $f'(k^*) > f'(k_{GR})$. We know that $f'(k^*) = \delta + p + Gg$ and $f'(k_{GR}) = n + g + \delta$, so is $\delta + p + Gg > n + g + \delta$? So that lifetime utility does not diverge we previously had to assume that $\beta = p - n - (1 - G)g > 0$. The value of β has to be greater than zero so that our discount factor $e^{-\beta t}$ makes sense and does not say that people can gain more utility from consumption by waiting to consume to the extent that utility would

be infinite if one waited long enough. The inequality $p - n - (1 - G)g > 0$ can be written as $p - n - g + Gg > 0$, and so we know that $p + Gg > n + g$ and that $\delta + p + Gg > n + g + \delta$, therefore the steady point E must be to the left of the golden-rule point where consumption is maximized.

We now know that it seems that the golden-rule point is unattainable because c is not stable there, and perhaps it should be remembered that the goal here is to maximize utility, not necessarily consumption at the steady state. We are now concerned with how the economy must move toward the steady state point E when it is given an initial value of k . The initial value of k , which is $k(0)$, shows the resources that the representative household starts out with, and so it is given and does not need to be determined. The initial value of c needs to be determined, given an initial value of k , so that the economy will converge to the steady point.

Let's assume that $k(0)$ is less than k^* , then *Figure 9.5* roughly shows how the economy would behave according to *Figure 8.3* for differing initial values of c .

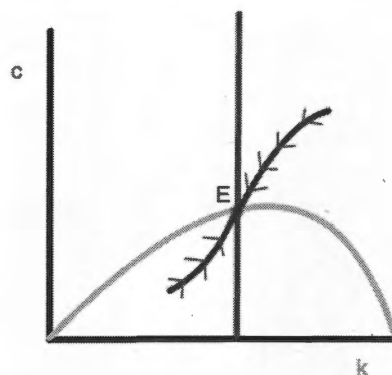


If the initial value of c is above the parabola where k is constant, the change in c is positive and the change in k is negative; the economy never converges to point E and this

situation does not follow the budget constraint, so the initial value of c cannot be in this region. If the initial value of c is too low, however, at first both c and k will be increasing, but k will cross the point where $k=k^*$ before point E is reached. After k becomes greater than k^* , k will still be increasing, but c will be decreasing, and although this path follows the budget constraint, it is not desirable because people will not maximize utility by acquiring an ever increasing amount of capital and lowering their consumption so much. Thus, we see that the initial level of capital that should be chosen must be at some point F than allows the economy to converge to the steady point E.

For every possible starting value of k , even if the starting value is greater than k^* , there is an initial value of c that will allow convergence to the steady state point. The initial value of c given as a function of the initial value of k is called the saddle path. A depiction of what the saddle path might look like can be found in *Figure 9.6*. Thus, a planner wishing to maximize the total utility of all the representative households must first choose the initial value of c that is on the saddle path when given the initial value of k . They then must follow the rule $\frac{\Delta c(t)}{c(t)} = \frac{r(t) - \delta - p - Gg}{G}$ to move along the saddle path to the steady state point E.

Figure 9.6



After point E is reached, we know that consumption per effective worker (c) and capital per effective worker (k) remain constant. Let y equal income per effective worker

$[y = \frac{Y}{AL}]$. Our production function is $Y = f(K, AL) = K^a(AL)^{1-a}$, so $y = \frac{Y}{AL} = \frac{K^a(AL)^{1-a}}{AL} =$

$K^a(AL)^{-a} = \left(\frac{K}{AL}\right)^a = k^a$. Because $y = k^a$, and because k remains constant at point E, we

know that there is no growth in income per effective worker (y) at the steady state.

However, because the term for technological change (A) is constantly growing at a rate of g , in order for $\frac{Y}{AL}$ to remain constant, $\frac{Y}{L}$ must also grow at a rate of g . Similarly, although

capital and consumption per effective worker remain constant at the steady state, capital

and consumption per worker must be growing at a rate of g as well. Thus, although there is

no growth in income or consumption per effective worker at the steady state in the

Ramsey-Cass-Koopmans model, there is growth in income and consumption per worker,

and it is occurring because of growth in technological change (Romer 49-65). Although the

Ramsey-Cass-Koopmans model allows the savings rate to be determined within the model,

this result is the same as a model where the savings rate is externally determined,

specifically the Solow growth model. These results suggest that instead of focusing on

making the savings rate endogenous, in order to better determine what causes income and

consumption per worker to increase, one should focus on making technological change

endogenous.

From the Ramsey-Cass-Koopmans model we have seen that there is a rule for the growth rate of consumption per effective worker that must be followed in order for total utility to be maximized. We have also seen that the economy needs to converge to some steady state point, that we called E, where the growth rates of consumption and capital per effective worker remain constant. Also, for any given amount of initial capital, an initial level of consumption must be chosen that falls on the saddle path so that the economy

converges to E . Finally, we have seen that although there is no growth in consumption per effective worker at the steady state point in this model, there is growth in consumption per worker. This growth in consumption per worker is driven by growth in the term " A " which was called "technological change", but is anything that makes labor more productive, thus focusing on what affects " A " will be important for other economic growth models.

Conclusion

Understanding what determines how much and when consumption takes place is crucial to predicting other important economic variables. Because of this, there have been many attempts to derive functions that show what factors influence consumption and how. Early versions of these functions, such as the Keynesian consumption function, were derived in a relatively simple manner, with just the help of intuition. Later, however, microeconomic ideas were incorporated to turn the process of determining consumption into an optimization problem. Looking at consumption as a problem of maximizing utility has led to important insights into the effects of interest and discount rates, and into how an entire economy might plan their consumption to maximize the utility of everyone. Although this paper does not discuss everything relating to the topic of consumption, as there are many other models and situations to consider, it provides a general overview of a few important theories of consumption and discusses how the general approach taken by economists has changed with regards to answering the questions relating to this topic.

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